Primer on Complex Numbers

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1 Imaginary numbers

For any real number $x \in \mathbb{R}$, $y = x^2 \geq 0$, independent of whether x is positive or negative. [Note $\in$ means "in" and $\mathbb{R}$ denotes the set of complex numbers.] This means $\sqrt{y}$ would only seem to be well-defined if $y \geq 0$. So we can ask the question: what is $\sqrt{-1}$? We know there's no real number x such that $x = \sqrt{-1}$, because $x^2 \geq 0$. So we need to extend the set of real numbers to include $\sqrt{-1}$. We call the $\sqrt{-1}$ the imaginary unit i, which satisfies

$$i^2 = -1$$

We'll assume that the standard rules of multiplication and addition apply to these imaginary numbers. So we can multiply the imaginary unit i by a real number to get another imaginary number. We say this number is imaginary because when we square it we get a negative real number, i.e. for $x \in \mathbb{R}$

$$(ix)^2 = i^2x^2 = -x^2$$

so if $r > 0$

$$\sqrt{-r} = \sqrt{r}\sqrt{-1} = \pm i\sqrt{r}.$$

2 Complex numbers

We saw above that imaginary numbers allow us to find $\sqrt{x}$ for any x, both $x \geq 0$ and $x < 0$. We could also define a **complex number** z as a number that consists of a sum of a real number a and an imaginary number ib (where $b \in \mathbb{R}$)

$$z = a + ib.$$

This set of numbers includes both the real numbers (when $b = 0$) and imaginary numbers (when $a = 0$). We'll assume that all of our standard rules of multiplication and addition carry over to complex numbers. So for $z = a + ib$, and $w = c + id$ addition can be carried out as

$$z + w = (a + ib) + (c + id) = (a + c) + (b + d)i$$

and multiplication can be done as follows

$$\begin{aligned} zw &= (a + ib)(c + id) \\ &= ac + ibc + iad + (ib)(id) \\ &= ac + i^2bd + i(bc + ad) \\ &= (ac - bd) + i(bc + ad) \end{aligned}$$

3 Absolute values

The **absolute value** or **modulus** of a complex number z, which we denote $|z|$ generalises tells us the size of a complex number. For $z = x + iy$ ($x, y \in \mathbb{R}$), $|z|$ is defined as

$$|z| = \sqrt{x^2 + y^2}$$

so $|z| \geq 0$. We always take the positive square root. This generalises the idea of the modulus/absolute value of a real number $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

4 Complex conjugate

The complex conjugate of a complex number $z = x + iy$ ($x, y \in \mathbb{R}$) is denoted z^*. The complex conjugate or simply **conjugate** is defined as

$$z^* = x - iy.$$

So we just change the sign of the imaginary part, and leave the real part unchanged.

It's fairly straight forward to verify that

$$(z + w)^* = z^* + w^*$$

and

$$(zw)^* = z^*w^*.$$

We can relate this to the modulus by considering z^*z

$$\begin{aligned} z^*z &= (x-iy)(x+iy) \\ &= x^2 + ixy - ixy + (iy)(-iy) \\ &= x^2 - (i^2)y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

So if we multiply any z by z^* we get a real number back. This can be useful for simplify expressions like w/z, for $z = a + ib$, and $w = c + id$

$$\begin{aligned} \frac{w}{z} &= \frac{w}{z}\frac{z}{z^*} \\ &= \frac{wz^*}{|z|^2} \\ &= \frac{(ca+bd)+(ad-bc)i}{a^2+b^2} \end{aligned}$$

5 Real and imaginary parts

The real part of a complex number $z = x + iy$ ($x, y \in \mathbb{R}$) is

$$\mathrm{Re}[z] = x$$

and likewise the imaginary part is

$$\mathrm{Im}[z] = y.$$

Using the rules of addition and multiplication it is fairly straightforward to show that

$$\mathrm{Re}[z] = \frac{1}{2}(z + z^*)$$

and

$$\mathrm{Im}[z] = -\frac{i}{2}(z - z^*).$$

We can relate the real and imaginary parts of a complex number by factoring our $|z|$

$$z = |z|\left(\frac{x}{|z|} + i\frac{y}{|z|}\right) = |z|\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right)$$

We also know that $x/|z|$ is bounded between -1 and 1 because

$$-1 \leq \frac{x}{\sqrt{x^2+y^2}} \leq 1$$

and likewise

$$-1 \leq \frac{y}{\sqrt{x^2+y^2}} \leq 1$$

so from this we know

$$-|z| \leq \mathrm{Re}[z] \leq |z|$$

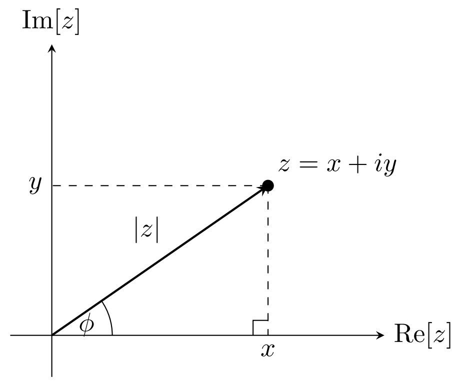
and

$$-|z| \leq \mathrm{Im}[z] \leq |z|$$

so the real and imaginary parts of a complex number are bounded below by $-|z|$ and above by $|z|$. These inequalities will be very useful in quantum mechanics.

6 Argand diagrams

We can map a complex number onto a point in 2D, with an x coordinate $x = \text{Re}[z]$ and $y = \text{Re}[z]$



We see that $|z|$ is the length of the line connecting the origin on this diagram to the complex number z (by Pythagoras theorem).

From the geometric view of the z on the Argand diagram, it's now very clear to see that

$$-|z| \leq \text{Re}[z] \leq |z|$$

and

$$-|z| \leq \text{Im}[z] \leq |z|.$$

7 Polar form of complex numbers

From the argand diagram we see there's another way we could write a complex number: in a **polar** coordinate form. ϕ is the angle made between the positive x ($\text{Re}[z]$) axis, and the line connecting the origin to the complex number z. As mentioned above $|z|$ is the length of the hypotenuse of the right-angled triangle connecting the origin, the point z, (x, y), and the point $(x, 0)$. Combining this we can write

$$\text{Re}[z] = x = |z|\cos(\phi)$$

and

$$\text{Im}[z] = y = |z|\sin(\phi).$$

This means we can instead write the complex number z in a polar form as

$$z = |z|(\cos(\phi) + i\sin(\phi)).$$

The angle ϕ is called the **argument** of the complex number $\arg(z)$

$$\arg(z) = \phi$$

and we normally take this to either be between 0 and 2π or $-\pi$ and π. Both conventions are fine to use and one may be more useful than the other in different circumstances.

8 Euler's formula

We will now show that polar form of a complex number can also be written as

$$z = |z|e^{i\phi}.$$

This requires us to show that **Euler's formula** is true

$$e^{ix} = \cos(x) + i\sin(x).$$

The Taylor series for e^x, $\cos(x)$ and $\sin(x)$ fully defined these functions. These Taylor series are given by

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

In order to evaluate e^{ix} we need $(ix)^{2n}$ and $(ix)^{2n+1}$

$$(ix)^{2n} = i^{2n}x^{2n} = (i^2)^n x^{2n} = (-1)^n x^{2n}$$

$$(ix)^{2n+1} = ix(ix)^{2n} = ix(-1)^n x^{2n} = i(-1)^n x^{2n+1}$$

Using the Taylor series for e^{ix}, we can split the sum into a sum over odd n and even n terms. We can write the odd terms as $n = 2m$ and even terms as $n = 2m+1$, and then remember that we can change the symbol for the index in the sum freely, so we replace $m \to n$. This gives

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}$$

Using the formulas we found above for $(ix)^{2n}$ and $(ix)^{2n+1}$ we can simplify this to

$$e^{ix} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

The two terms in this are just the Taylor series for $\cos(x)$ and $i\sin(x)$, so we have found

$$e^{ix} = \cos(x) + i\sin(x).$$

This proves Euler's formula.

9 Properties of the polar form

The polar form is very useful for simplifying products of complex numbers. We can write $z_1 = |z_1|e^{i\phi_1}$ and $z_2 = |z_2|e^{i\phi_2}$, and using $e^a e^b = e^{a+b}$ we find

$$z_1 z_2 = |z_1||z_2|e^{i(\phi_1+\phi_2)}$$

Likewise $|z|^* = |z|$, because $|z|$ is just a real number, and for real ϕ

$$(e^{i\phi})^* = (\cos(\phi) + i\sin(\phi))^* = \cos(\phi) - i\sin(\phi) = \cos(-\phi) + i\sin(-\phi) = e^{-i\phi}$$

so

$$z^* = |z|e^{-i\phi}.$$

Other formulae that are often useful are

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$

and

$$\sin(x) = -\frac{i}{2}(e^{ix} - e^{-ix}).$$

This is analogous to the hyperbolic tangent functions

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$$

and

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}).$$

10 Exercises

Try these short exercises to test your understanding.

1. Simplify the following complex numbers (i) $(2+i)(3-2i)$ (ii) $(2+i)/(3-2i)$ (iii) $e^{i(3\pi/2)}$
2. Show that $e^z = e^{\mathrm{Re}[z]}(\cos(\mathrm{Im}[z]) + i\sin(\mathrm{Im}[z]))$.
3. Show that $|e^{i\phi}| = 1$, and $|e^{i\phi}z| = |z|$ for $\phi \in \mathbb{R}$.
4. Use the polar form of complex numbers z_1 and z_2 to write down z_1/z_2 in polar form.
5. Derive expressions for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin(\alpha)$, $\sin(\beta)$, $\cos(\alpha)$ and $\cos(\beta)$ by considering $e^{i\alpha}e^{i\beta}$.