Primer on Fourier Transforms and the Delta function

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1 Fourier Series

In the homework problems you showed that for periodic functions $f(x) = f(x + 2\pi)$, we can write the function $f(x)$ as

$$f(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imx} \tilde{f}_m$$

where $\tilde{f}_m$ is the Fourier coefficient

$$\tilde{f}_m = \int_{-\pi}^{\pi} e^{-imx} f(x) \, \mathrm{d}x \, .$$

It is straightforward from this to show that $2L$ periodic functions $f(x) = f(x + 2L)$ can be written as

$$f(x) = \frac{1}{2\pi} \left(\frac{\pi}{L}\right) \sum_{m=-\infty}^{\infty} e^{i\pi mx/L} \tilde{f}_m$$

where the Fourier coefficient is

$$\tilde{f}_m = \int_{-L}^{L} e^{-i\pi mx/L} f(x) \, \mathrm{d}x \, .$$

writing $k_m = \pi m/L$ we can write the Fourier coefficient as a function $\tilde{f}(k_m)$

$$\tilde{f}_m \equiv \tilde{f}(k_m) = \int_{-L}^{L} e^{-ik_m x} f(x)\,\mathrm{d}x\,.$$

and we can therefore write $f(x)$ as

$$f(x) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \Delta k e^{ik_m x}\tilde{f}(k_m)$$

where we have rather suggestively written $\Delta k = \pi/L$.

We now take the limit $L \to \infty$, in which case $\tilde{f}(k)$ becomes the **Fourier transform** of $f(x)$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x)\,\mathrm{d}x\,.$$

and the **inverse Fourier transform** is given by

$$\begin{aligned} f(x) &= \lim_{L\to\infty}\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \Delta k e^{ik_m x}\tilde{f}(k_m) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,\mathrm{d}k \end{aligned}$$

where we just used the definition of the (Riemann) integral.

This is not a very rigorous derivation of the Fourier transform, but it will suffice. We need to be careful to make sure that $\tilde{f}(k)$ remains well defined as $L \to \infty$ so we generally need $|f(x)| \to 0$ as $|x| \to \infty$.

The Fourier transform can be thought of as a decomposition of $f(x)$ into components oscillating at different (angular) frequencies k, in much the same way as the Fourier series for periodic functions.

In the language of Linear Algebra we can write $|u(k)\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$, and therefore we can write

$$\begin{aligned} \tilde{f}(k) &= \sqrt{2\pi}\,\langle u(k)|f\rangle \\ |f\rangle &= \int_{-\infty}^{\infty} \mathrm{d}k\,|u(k)\rangle\,\langle u(k)|f\rangle \end{aligned}$$

so this implies we can write the identity operator as

$$\hat{1} = \int_{-\infty}^{\infty} \mathrm{d}k\,|u(k)\rangle\langle u(k)| \tag{1}$$

So the vectors $|u(k)\rangle$ can be said to form a basis. Note however that because the basis functions $|u(k)\rangle$ are continuous functions of k (and not discrete as we had with periodic functions) we have to be careful when evaluating the overlap $\langle u(k)|u(k')\rangle$. We will revisit this shortly.

2 Properties of the Fourier transform

The Fourier transform has many useful properties. Perhaps one of the most widely used is, that if $g(x) = \frac{\mathrm{d}}{\mathrm{d}x}f(x)$ then the Fourier transform is given by

$$\tilde{g}(k) = ik\tilde{f}(k). \tag{2}$$

The proof follows either using integration by parts or differentiation under the integral of $g(x)$ using the inverse Fourer transform. By induction we find that if $g(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)$ then the Fourier transform is

$$\tilde{g}(k) = (ik)^n\tilde{f}(k) \tag{3}$$

Likewise if $g(x) = x^n f(x)$ then

$$\tilde{g}(k) = i^n \frac{\mathrm{d}^n}{\mathrm{d}k^n} \tilde{f}(k). \tag{4}$$

Lots of other useful properties can be found just using elementary rules of integration.

3 The Delta function

Using the definitions of the Fourier transform and its inverse we find that

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k)\, \mathrm{d}k \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx} \int_{-\infty}^{\infty} \mathrm{d}x'\, e^{-ikx'} f(x') \end{aligned}$$

Swapping the order of integration we find

$$f(x) = \int_{-\infty}^{\infty} \mathrm{d}x'\, f(x') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, e^{ik(x-x')} \right)$$

The term in brackets is just a function of $x - x'$, and when integrated with $f(x')$ returns $f(x)$. We call this the **Dirac δ function**.

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx} \tag{5}$$

It has the **sifting** property that

$$f(x) = \int_{-\infty}^{\infty} \mathrm{d}x'\, f(x') \delta(x - x'). \tag{6}$$

If we integrate the above equation with respect to x we find that

$$1 = \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x - x'). \tag{7}$$

and from the Fourier definition $\delta(x) = \delta(-x)$ so it is an even function. By interpreting it as a probability distribution we also find that

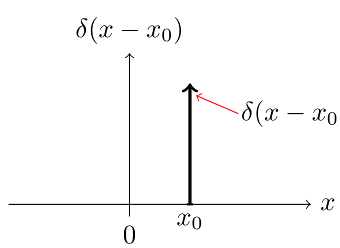
$$x' = \int_{-\infty}^{\infty} \mathrm{d}x\, x\ \delta(x - x'). \tag{8}$$

so its average is x' and we find the central moments to be

$$0 = \int_{-\infty}^{\infty} \mathrm{d}x\, (x - x')^n\ \delta(x - x'). \tag{9}$$

for $n > 0$. Note that the $n = 2$ moment is the variance which is zero. This suggests that $\delta(x)$ can be interpreted as a distribution with no variance whatsoever, and an average value of 0.

We conclude the $\delta(x)$ is zero everywhere apart from at $x = 0$ where it diverges.



This means when integrating $f(x)\delta(x)$ only the value of $f(x)$ at $x = 0$ is picked out. This hopefully gives some intuition for what the delta function "looks like".

There are many possible ways to define the delta function as limits of other functions. For example we can define it in terms of a Gaussian integral

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \tag{10}$$

or an exponential

$$\delta(x) = \lim_{a \to \infty} \frac{a}{2} e^{-a|x|}. \tag{11}$$

4 Properties of the delta function

A lot of basic properties of the delta function can be found from properties of intergals. For example the derivative of the delta function gives

$$\int_{-\infty}^{\infty} \mathrm{d}x' \, f(x') \frac{\mathrm{d}}{\mathrm{d}x'} \delta(x' - x) = [f(x')\delta(x' - x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathrm{d}x' \, f'(x')\delta(x' - x) \tag{12}$$

$$= -f'(x) \tag{13}$$

By induction we have

$$\int_{-\infty}^{\infty} \mathrm{d}x' \, f(x') \frac{\mathrm{d}^n}{\mathrm{d}x'^n} \delta(x' - x) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) \tag{14}$$

Also we can consider integration over a finite domain of $\delta(x)$

$$\int_a^b \delta(x - x_0) f(x) \, \mathrm{d}x = \begin{cases} f(x_0), & \text{if } a < x_0 < b \\ f(x_0)/2, & \text{if } x_0 = a \text{ or } x_0 = b \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

The $f(x_0)/2$ at the boundary ensures we can stitch together integration domains and still get the right answer. This will only be non-zero of the peak of the delta-function x_0 is inside the integration domain.

5 Heaviside step function

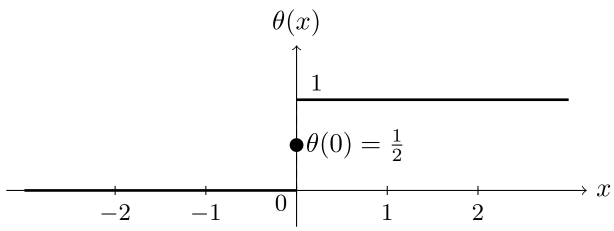
In particular we can write

$$\int_{-\infty}^{x} \mathrm{d}x' \, \delta(x') = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{16}$$

This is the definition of $\theta(x)$, the **Heaviside** function, also known as the **step** function, so called because it steps from 0 to 1 between negative and positive x.

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{17}$$

This is sometimes denoted as $h(x)$ or $\Theta(x)$ as well.



From this definition we see that the derivative of the Heaviside function is the delta function

$$\frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \delta(x). \tag{18}$$

6 Normalisation of $|u(k)\rangle$

We can now revisit the normalisation of the vectors $|u(k)\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$. By direct evaluation of the inner produce we find

$$\langle u(k)|u(k')\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ikx}e^{ik'x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{i(k'-k)x} \tag{19}$$

$$= \delta(k'-k) = \delta(k-k') \tag{20}$$

So when we have a continuous basis like $|u(k)\rangle$ we normalise the vectors such that $\langle u(k)|u(k')\rangle = \delta(k-k')$. So the delta function can be thought of as the continuous analogue of the Kronecker delta $\delta_{n,m}$.